O.R. Applications

Matching daily healthcare provider capacity to demand in advanced access scheduling systems

Xiuli Qu a,*, Ronald L. Rardin a, Julie Ann S. Williams b, Deanna R. Willis c

a Purdue University, School of Industrial Engineering, 315 N. Grant Street, West Lafayette, IN 47907-2023, United States
b University of West Florida, Department of Management and MIS, 11000 University Parkway, Bldg 76128, Pensacola, FL 32514-5752, United States
c Indiana University, Department of Family Medicine, 1110 W Michigan Street, Indianapolis, IN 46202, United States

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Abstract

Advanced access scheduling, introduced in the early 1990s, is reported to significantly improve the performance of outpatient clinics. The successful implementation of advanced access scheduling requires the match of daily healthcare provider capacity with patient demand. In this paper, for the first time a closed-form approach is presented to determine the optimal percentage of open-access appointments to match daily provider capacity to demand. This paper introduces the conditions for the optimal percentage of open-access appointments and the procedure to find the optimal percentage. Furthermore, the sensitivity of the optimal percentage of open-access appointments to provider capacity, no-show rates, and demand distribution is investigated. Our results demonstrate that the optimal percentage of open-access appointments mainly depends on the ratio of the average demand for open-access appointments to provider capacity and the ratio of the show-up rates for prescheduled and open-access appointments.

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1. Introduction

Currently, inefficiency of healthcare delivery and barrier to access healthcare are two critical problems faced by most outpatient clinics in the United States. The traditional appointment scheduling systems currently used in outpatient clinics are one of the main causes for the two problems. At the end of a patient’s current visit, the traditional appointment scheduling systems schedule his/her next visit months in advance. If patients call for a non-urgent visit, they have to wait from several weeks to several months for their visits. The long wait for visits damages timely care. On the other hand, if emergency care is used when patients cannot be seen by their own physicians in time, continuity of care is broken. Loss of timely care or continuity of care undermines
A new appointment scheduling concept for outpatient clinics, labeled advanced access scheduling, open access scheduling, or same-day scheduling, was introduced in the early 1990s to schedule patients within 12–72 hours of their requests regardless of the reasons for their visits (Herriott, 1999; Murray and Tantau, 2000). The key principle of the advanced access scheduling is to see patients when they want to be seen, similar to the idea of the just-in-time model in the manufacturing industry. It is reported that advanced access scheduling has been successfully implemented in many primary care practices, several family care practices, a residency training practice, and a group practice network. The successful implementation of advanced access scheduling is reported to improve the continuity of care, clinician resource utilization, and patient satisfaction while reducing healthcare costs (Herriott, 1999; Murray and Tantau, 2000; Kennedy and Hsu, 2003; Kodjababian, 2003; Murray et al., 2003; O’Hare and Corlett, 2004; Pierdon et al., 2004). However, the successful implementation of advanced access scheduling in an outpatient clinic requires a challenging transition and the match of healthcare provider capacity and patient demand on a daily basis (Kodjababian, 2003; Murray and Berwick, 2003).

Although operations research methods, such as queuing theory, stochastic optimization, and simulation, are applied to analyzing and optimizing traditional appointment scheduling systems (Brahimi and Worthington, 1991; Ho and Lau, 1992; Klassen and Rohleder, 1996; Denton and Gupta, 2003), critical parameters in advanced access scheduling systems are still determined by an expert’s experience rather than quantitative methods. For example, in advanced access scheduling systems, some appointments, usually called open-access appointments or same-day appointments, can be scheduled only within 12–72 hours. The percentage of open-access appointments in a clinic session is one of the critical parameters for matching daily provider capacity with patient demand (Herriott, 1999). In the literature, the percentage of open-access appointments in a session may range from 30% to 80% depending on the experiences of the expert managing the implementation of the advanced access scheduling system (Herriott, 1999; Murray and Tantau, 2000; Kennedy and Hsu, 2003). Dr. Mark Murray, one of the proponents of advanced access scheduling, recommends approximately 65% to 75% of open-access appointments (Murray and Tantau, 2000), but emphasizes that the appropriate percentage of open-access appointments with a provider depends on his/her patient population, level of patient demand for visits and his/her capacity to offer appointments (Murray and Berwick, 2003). Herriott (1999) reports that according to an expert’s recommendation, 50% of total appointments are held open in a clinic on the weekdays with high demand, and 30–35% of total appointments are held open in the clinic on the weekdays with low demand. It is reported that more than 80% of appointments are held open in a residency training clinic (Kennedy and Hsu, 2003).

Inappropriate percentages of open-access appointments with a provider in his/her clinic sessions will result in the mismatch of his/her capacity and patient demand and ultimately an implementation failure for advanced access scheduling in a clinic. Therefore, quantitative methods need to be developed for clinical administrators to determine the optimal percentage of open-access appointments available with each provider in each clinic session to match daily capacity with demand at the provider level. In the literature, no quantitative methods for optimizing the performance of advanced access scheduling systems have been reported to the authors’ knowledge. However, we briefly review the quantitative methods to optimize the performance of traditional outpatient appointment scheduling systems. Queuing theory and simulation are the first quantitative methods used to optimize and compare the performance of outpatient scheduling systems. Using queuing theory, Jansson (1966) studies the joint distribution of patient waiting time and provider idle time assuming that the provider and patients arrive punctually, and then optimizes the scheduling rule based on the joint distribution. Soriano (1966) compares two scheduling policies using a single-server queuing model with batch arrivals. Pegden and Rosenshine (1990) and Brahimi and Worthington (1991) propose more general queuing models to determine better scheduling policies. In early studies, simulation is used to determine a better scheduling rule by comparing the performance of several scheduling rules (Bailey, 1952; Blanco White and Pike, 1964; Fetter and Thompson, 1966). In later studies, simulation experiments are conducted and analyzed to find the most appropriate scheduling rule for different outpatient clinics (Ho and Lau, 1992; Klassen and Rohleder, 1996). Recently, dynamic programming is used to optimize the number of patients scheduled in each appointment slot for outpatient clinics (Fries and Marathe,
1981; Liao et al., 1993). Other optimization methods are employed to optimize appointment scheduling systems for other medical services. For example, Denton and Gupta (2003) and Robinson and Chen (2003) optimize scheduling for operating rooms using stochastic linear programming and linear programming with Monte Carlo sampling, respectively. A comprehensive review on traditional outpatient appointment scheduling can be found in Cayirli and Veral (2003).

In this paper, we present a quantitative approach to determine the optimal percentage of appointments held open in a session. In the next section, we briefly describe advanced access scheduling systems, and define and formulate the problem. The conditions for the optimal solution to the problem and the procedure to find the optimal solution are presented in Sections 3 and 4, respectively. In Section 5, numerical examples are solved to test the sensitivity of the optimal percentage of open-access appointments to the demand distribution and no-show rate, which is defined as the ratio of the number of appointments missed to the number of appointments scheduled. Finally, potential future research is discussed and conclusions are drawn in Section 6.

2. Problem definition and formulation

In an advanced access scheduling system, some appointments, which are called prescheduled appointments in this paper, can be scheduled months in advance. However, the other appointments, which are called open-access appointments in this paper, can only be scheduled within 12–72 hours. Due to the clinical practices, providers’ schedules, and patient population, the time interval in which open-access appointments can be scheduled varies among the clinics implementing advanced access scheduling. Usually, open-access appointments can be scheduled within 24 hours or same day. However, in some clinics, open-access appointments can be scheduled within 72 hours or even one week (South Texas Veterans Healthcare Care System, 2004). In advanced access scheduling systems, clinic administrators limit the maximal number/percentage of appointments that can be prescheduled in a clinic session or the minimal number/percentage of appointments held open until 12–72 hours of a session. When the number of appointments prescheduled in a session reaches the limit, patients who request prescheduled appointments in the session have to choose prescheduled appointments in other sessions or call again later for an open-access appointment in the session.

For a clinic, the average number of patients consulted by a provider in a session is a measure of its productivity, and is also one of the measures for the match of daily demand and capacity. Therefore, the optimal percentage of open-access appointments with a provider in a session needs to maximize the expected number of patients consulted by the provider in the session. Since the no-show rate increases with the increase in the length of interval from the date an appointment is scheduled to the appointment date (Kodjababian, 2003), a high percentage of prescheduled appointments results in a risk of more no-shows. On the other hand, a high percentage of open-access appointments risks fewer appointments scheduled if there is not enough demand for open-access appointments. Both risks decrease the expected number of patients consulted by a provider in a session. As a result, the optimal percentage of open-access appointments in a clinic session needs to be determined for each provider to balance the two risks. The assumptions, definition and formulation of the problem are presented in Sections 2.1, 2.2 and 2.3, respectively.

2.1. Assumptions

In this paper, we assume that in a clinic the appointment scheduling for one provider is independent of the appointment scheduling for other providers. Thus the optimal percentage of open-access appointments can be determined for each provider regardless of the appointment scheduling policies for other providers. We also assume that the attendance of a patient is independent of the attendance of other patients.

In addition, it is assumed that for a given provider, the total number of appointments that can be scheduled in a session is fixed, the distribution of demands for prescheduled and open-access appointments is known, and the no-show rates of prescheduled and open-access appointments are also known, denoted by $\gamma_1$ and $\gamma_2$, respectively. Since the no-show rate of prescheduled appointments is usually higher than the no-show rate of open-access appointments (Kodjababian, 2003; O’Hare and Corlett, 2004), it is assumed that $\gamma_1 > \gamma_2$.
2.2. Problem definition

Let \( D_1 \) and \( D_2 \) denote the random demands for prescheduled and open-access appointments with a provider in a session, respectively. The joint probability mass function of \( D_1 \) and \( D_2 \) is
\[
p(d_1, d_2) = P(D_1 = d_1, D_2 = d_2) \quad \text{for } d_1 = 0, 1, 2, \ldots \text{ and } d_2 = 0, 1, 2, \ldots .
\]
Since the total number of appointments available with a provider in a session, denoted by \( N \), is fixed, determining the optimal percentage of open-access appointments with a provider in a session is equivalent to determining the optimal number of appointments that can be prescheduled with the provider in a session. Thus the problem can be defined as

Given \( p(d_1, d_2) \), \( \gamma_1 \), \( \gamma_2 \), and \( N \), find the optimal number of appointments that can be prescheduled, \( N_1^* \), prior to scheduling any appointment for the provider in the session, such that the expected number of patients consulted is maximized.

2.3. Formulation

Let \( Q(N_1) \) denote the expected number of patients consulted by a provider in a clinic session if at most \( N_1 \) appointments can be prescheduled with the provider in the session. The expected number of patients consulted by a provider in a session equals the sum of the expected numbers of prescheduled and open-access appointments kept. Let \( Q_1(N_1) \) and \( Q_2(N_1) \) denote the expected numbers of prescheduled and open-access appointments kept, respectively, if at most \( N_1 \) appointments can be prescheduled with the provider in the session. Since
\[
Q_1(N_1) = (1 - \gamma_1)E[\min(D_1, N_1)],
\]
and
\[
Q_2(N_1) = (1 - \gamma_2)E[\min(D_2, N - \min(D_1, N_1))],
\]
we have
\[
Q(N_1) = Q_1(N_1) + Q_2(N_1) = (1 - \gamma_1)E[\min(D_1, N_1)] + (1 - \gamma_2)E[\min(D_2, N - \min(D_1, N_1))].
\]
Then the problem defined in Section 2.2, can be formulated as
\[
\begin{align*}
\text{maximize} & \quad Q(N_1) = (1 - \gamma_1)E[\min(D_1, N_1)] + (1 - \gamma_2)E[\min(D_2, N - \min(D_1, N_1))] \\
\text{subject to} & \quad N_1 \leq N, \\
& \quad N_1 \text{ is integer}.
\end{align*}
\]

3. Conditions for the optimal solution

In Section 2, we defined and formulated the problem. For a given \( N \), the distribution of demands, and the no-show rates of prescheduled and open-access appointments, the optimal number of appointments that can be prescheduled with a provider in a session, \( N_1^* \), can be determined by solving Problem \((P_1)\). In this section, the conditions for the optimal solution, \( N_1^* \), to Problem \((P_1)\) are derived and interpreted. First, we derive the expected number of patients consulted by a provider in a clinic session.

3.1. Expected number of patients consulted

Let \( p_1(d_1) = P(D_1 = d_1) \), for \( d_1 \in [0, \infty) \), denote the probability mass function of demand \( D_1 \). The expected number of appointments prescheduled with a provider in a session is
\[
E[\min(D_1, N_1)] = \sum_{d_1=0}^{N_1} d_1 p_1(d_1) + \sum_{d_1=N_1+1}^{\infty} N_1 p_1(d_1) = N_1 - \sum_{d_1=0}^{N_1} (N_1 - d_1) p_1(d_1),
\]
and the expected number of open-access appointments scheduled with a provider in a session is

\[
E[\min(D_2, N - \min(D_1, N_1))] = \sum_{d_1=0}^{N_1} \left[ \sum_{d_2=0}^{N-d_1} d_2p(d_1, d_2) + \sum_{d_2=N-d_1+1}^{\infty} (N - d_1)p(d_1, d_2) \right] \\
+ \sum_{d_1=N_1+1}^{\infty} \left[ \sum_{d_2=0}^{N-N_1} d_2p(d_1, d_2) + \sum_{d_2=N-N_1+1}^{\infty} (N - N_1)p(d_1, d_2) \right] \\
= \sum_{d_1=0}^{N_1} \left[ (N - d_1)p_1(d_1) - \sum_{d_2=0}^{N-d_1} (N - d_1 - d_2)p(d_1, d_2) \right] \\
+ \sum_{d_1=N_1+1}^{\infty} \left[ (N - N_1)p_1(d_1) - \sum_{d_2=0}^{N-N_1} (N - N_1 - d_2)p(d_1, d_2) \right] \\
= (N - N_1) + \sum_{d_1=0}^{N_1} (N_1 - d_1)p_1(d_1) - \sum_{d_1=0}^{N_1} \sum_{d_2=0}^{N-d_1} (N - d_1 - d_2)p(d_1, d_2) \\
- \sum_{d_1=N_1+1}^{\infty} \sum_{d_2=0}^{N-N_1} (N - N_1 - d_2)p(d_1, d_2). \tag{7}
\]

Substituting Eqs. (6) and (7) in Eq. (3), we obtain the expected number of patients consulted by a provider in a session

\[
Q(N_1) = -\gamma_1 - \gamma_2 \left[ N_1 - \sum_{d_1=0}^{N_1} (N_1 - d_1)p_1(d_1) \right] \\
+ (1 - \gamma_2) \left[ N - \sum_{d_1=0}^{N_1} \sum_{d_2=0}^{N-d_1} (N - d_1 - d_2)p(d_1, d_2) - \sum_{d_1=N_1+1}^{\infty} \sum_{d_2=0}^{N-N_1} (N - N_1 - d_2)p(d_1, d_2) \right]. \tag{8}
\]

### 3.2. Dependent demands for prescheduled and open-access appointments

If the demands for prescheduled and open-access appointments are dependent on each other, Theorem 1 presents the conditions for the optimal solution, \(N_1^*\), to Problem \((P_1)\).

**Theorem 1.** A local optimal solution to Problem \((P_1)\) is an integer \(N_1 \in \{1, 2, \ldots, N - 1\}\) such that

\[
P(D_2 > N - N_1 | D_1 > N_1 - 1) \leq \frac{1 - \gamma_1}{1 - \gamma_2}, \tag{9}
\]

and

\[
P(D_2 > N - N_1 - 1 | D_1 > N_1) \geq \frac{1 - \gamma_1}{1 - \gamma_2}. \tag{10}
\]

If there exists no \(N_1 \in \{1, 2, \ldots, N - 1\}\) satisfying conditions (9) and (10), the optimal solution should be either 0 or \(N\). If condition (9) holds for all \(N_1 \in \{1, 2, \ldots, N\}\), the optimal solution to Problem \((P_1)\) is \(N\). On the contrary, if condition (10) holds for all \(N_1 \in \{0, 1, 2, \ldots, N - 1\}\), the optimal solution is 0.

**Proof.** Since \(N_1 \in \{0, 1, 2, \ldots, N\}\), if there exists an integer \(i \in \{1, 2, \ldots, N - 1\}\) satisfying \(Q(i) - Q(i - 1) \geq 0\) and \(Q(i + 1) - Q(i) \leq 0\), then \(i\) is a local optimal solution to Problem \((P_1)\).

According to Eq. (A.1) in the Appendix, we know

\[
Q(i) - Q(i - 1) = (1 - \gamma_1)[1 - F_1(i - 1)] - (1 - \gamma_2)[1 - F_1(i - 1) - F_2(N - i) + F(i - 1, N - i)], \tag{11}
\]
and
\[ Q(i + 1) - Q(i) = (1 - \gamma_1)[1 - F_1(i)] - (1 - \gamma_2)[1 - F_1(i) - F_2(N - i - 1) + F(i, N - i - 1)], \]

where \( F_1(i) = P(D_1 \leq i) \) is the cumulative probability distribution function of demand \( D_1 \), \( F_2(i) = P(D_2 \leq i) \) is the cumulative probability distribution function of demand \( D_2 \), and \( F(i, j) = P(D_1 \leq i, D_2 \leq j) \) is the joint cumulative probability distribution function of demands \( D_1 \) and \( D_2 \). Thus, inequalities (13) and (14) are equivalent to \( Q(i) - Q(i - 1) \geq 0 \) and \( Q(i + 1) - Q(i) \leq 0 \), respectively.

\[ \frac{1 - F_1(i - 1) - F_2(N - i) + F(i - 1, N - i)}{1 - F_1(i - 1)} \leq \frac{1 - \gamma_1}{1 - \gamma_2}, \]

(13)

\[ \frac{1 - F_1(i) - F_2(N - i - 1) + F(i, N - i - 1)}{1 - F_1(i)} \geq \frac{1 - \gamma_1}{1 - \gamma_2}. \]

(14)

Since \( P(D_1 > i - 1, D_2 > N - i) = 1 - F_1(i - 1) - F_2(N - i) + F(i - 1, N - i) \) and \( P(D_1 > i - 1) = 1 - F_1(i - 1) \), we therefore obtain

\[ \frac{1 - F_1(i - 1) - F_2(N - i) + F(i - 1, N - i)}{1 - F_1(i - 1)} = P(D_2 > N - i | D_1 > i - 1). \]

Similarly, we have

\[ \frac{1 - F_1(i) - F_2(N - i - 1) + F(i, N - i - 1)}{1 - F_1(i)} = P(D_2 > N - i - 1 | D_1 > i). \]

Thus, the inequalities (15) and (16) are also equivalent to \( Q(i) - Q(i - 1) \geq 0 \) and \( Q(i + 1) - Q(i) \leq 0 \), respectively.

\[ P(D_2 > N - i | D_1 > i - 1) \leq \frac{1 - \gamma_1}{1 - \gamma_2}, \]

(15)

\[ P(D_2 > N - i - 1 | D_1 > i) \geq \frac{1 - \gamma_1}{1 - \gamma_2}. \]

(16)

Therefore, if there exists an integer \( i \in \{1, 2, \ldots, N - 1\} \) satisfying inequalities (15) and (16), the integer \( i \) is a local optimal solution to Problem \( (P_1) \). The first part of Theorem 1 is proven.

If there exists no integer \( i \in \{1, 2, \ldots, N - 1\} \) satisfying \( Q(i) - Q(i - 1) \geq 0 \) and \( Q(i + 1) - Q(i) \leq 0 \), then either for all \( i \in \{1, 2, \ldots, N\} \) \( Q(i) - Q(i - 1) \geq 0 \) holds or for all \( i \in \{0, 1, 2, \ldots, N - 1\} \) \( Q(i + 1) - Q(i) \leq 0 \) holds.

If for all \( i \in \{1, 2, \ldots, N\} \), \( Q(i) - Q(i - 1) \geq 0 \) holds, which means \( Q(0) \leq Q(1) \leq \cdots \leq Q(N - 1) \leq Q(N) \), then the optimal solution to Problem \( (P_1) \) is \( N \). Since inequality (15) is equivalent to \( Q(i) - Q(i - 1) \geq 0 \), if for all \( i \in \{1, 2, \ldots, N\} \), inequality (15) holds, then the optimal solution is \( N \).

On the contrary, if for all \( i \in \{0, 1, 2, \ldots, N - 1\} \), \( Q(i + 1) - Q(i) \leq 0 \) holds, which means \( Q(0) \geq Q(1) \geq \cdots \geq Q(N - 1) \geq Q(N) \), then the optimal solution to Problem \( (P_1) \) is 0. Since inequality (16) is equivalent to \( Q(i + 1) - Q(i) \leq 0 \), if for all \( i \in \{0, 1, 2, \ldots, N - 1\} \), inequality (16) holds, then the optimal solution is 0. \( \square \)

3.3. Independent demands for prescheduled and open-access appointments

Theorem 1 provides the conditions for the optimal solution to Problem \( (P_1) \) regardless of whether the demands for prescheduled and open-access appointment, \( D_1 \) and \( D_2 \), are dependent or independent of each other. If \( D_1 \) and \( D_2 \) are independent of each other, the conditions for the optimal solution are given by Corollary 1.

**Corollary 1.** If the random demands for prescheduled and open-access appointments, \( D_1 \) and \( D_2 \), are independent of each other, a local optimal solution to Problem \( (P_1) \) is an integer \( N_1 \in \{1, 2, \ldots, N - 1\} \) such that

\[ P(D_2 > N - N_1) \leq \frac{1 - \gamma_1}{1 - \gamma_2}, \]

(17)
and
\[ P(D_2 > N - N_1 - 1) \geq \frac{1 - \gamma_1}{1 - \gamma_2}. \]  \hfill (18)

If there exists no \( N_1 \in \{1, 2, \ldots, N - 1\} \) satisfying conditions (17) and (18), the optimal solution should be either 0 or \( N \). If condition (17) holds for all \( N_1 \in \{1, 2, \ldots, N\} \), the optimal solution is \( N \). On the contrary, if condition (18) holds for all \( N_1 \in \{0, 1, 2, \ldots, N - 1\} \), the optimal solution is 0.

**Proof.** For independent demands \( D_1 \) and \( D_2 \), we have
\[ P(D_2 > N - N_1|D_1 > N_1 - 1) = P(D_2 > N - N_1), \]
and
\[ P(D_2 > N - N_1 - 1|D_1 > N_1) = P(D_2 > N - N_1 - 1). \]

Thus, inequalities (17) and (18) are equivalent to inequalities (9) and (10) in Theorem 1, respectively. According to Theorem 1, the corollary is proven. \( \square \)

3.4. Interpretation of the conditions for the optimal solution

In the conditions for the optimal solution, \( N_1^* \), to Problem (P1), \( (1 - \gamma_1)/(1 - \gamma_2) \) represents the ratio of the show-up rate of prescheduled appointments to the show-up rate of open-access appointments, where the show-up rate is defined as the ratio of the number of appointments kept to the number of appointments scheduled. If the demands for prescheduled and open-access appointments with a provider in a clinic session are dependent on each other, the optimal solution, \( N_1^* \), depends on the total number of appointments available with the provider in the session, the joint distribution of the two demands, and the ratio of the two show-up rates. If the two demands are independent of each other, the optimal solution, \( N_1^* \), does not depend on the distribution of the demand for prescheduled appointments.

For the dependent demands for prescheduled and open-access appointments, the two conditions for the optimal \( N_1 \) can be rewritten as
\[ (1 - \gamma_2)P(D_1 \geq N_1, D_2 > N - N_1) \leq (1 - \gamma_1)P(D_1 \geq N_1), \]  \hfill (19)

and
\[ (1 - \gamma_2)P(D_1 > N_1, D_2 \geq N - N_1) \geq (1 - \gamma_1)P(D_1 > N_1). \]  \hfill (20)

In condition (19), \( P(D_1 \geq N_1, D_2 > N - N_1) \) represents the probability that all prescheduled and open-access appointments available are filled and \( P(D_1 \geq N_1) \) represents the probability that all prescheduled appointments available are filled if at most \( N_1 \) prescheduled appointments are available with a provider in a clinic session. Thus, when the number of prescheduled appointments available with the provider in a clinic session increases from \( N_1 - 1 \) to \( N_1 \), the right-hand side in condition (19) represents the increase in the expected number of patients consulted by the provider in a clinic session, while the left-hand side of condition (19) represents the decrease in the expected number of patients consulted. Similarly, in condition (20), \( P(D_1 > N_1, D_2 \geq N - N_1) \) represents the probability that all prescheduled and open-access appointments available are filled and \( P(D_1 > N_1) \) represents the probability that all prescheduled appointments available are filled if at most \( N_1 + 1 \) prescheduled appointments are available. Thus, when the number of prescheduled appointments available with the provider in a clinic session increases from \( N_1 \) to \( N_1 + 1 \), the right-hand side in condition (20) represents the increase in the expected number of patients consulted while the left-hand side of condition (20) represents the decrease in the expected number of patients consulted. Therefore, at the optimal value for \( N_1 \), the gain and loss of the expected number of patients consulted are very close when \( N_1 \) changes by one.
4. Procedure to find the optimal solution

In Section 3, the formula for the expected number of patients consulted by a provider in a clinic session, \( Q(N_1) \), is given in Eq. (8). However, the formula is not computationally convenient. According to Eq. (A.1) in the Appendix, we have the recurrence relationship in Eq. (21) to quickly compute \( Q(N_1) \) for \( 0 < N_1 \leq N \).

\[
Q(N_1) = Q(N_1 - 1) - (\gamma_1 - \gamma_2)[1 - F_1(N_1 - 1)] + (1 - \gamma_2)[F_2(N - N_1) - F(N_1 - 1, N - N_1)].
\]  

(21)

According to Eq. (8), when \( N_1 = 0 \), the expected number of patients consulted by a provider in a session is

\[
Q(0) = (1 - \gamma_2) \left[ N - \sum_{d_2=0}^{N} (N - d_2)p_2(d_2) \right].
\]  

(22)

Using Eqs. (21) and (22), the expected number of patients consulted by a provider in a clinic session, \( Q(N_1) \) for \( 0 \leq N_1 \leq N \), can be computed using Procedure 1 as follows.

Procedure 1– Expected number of patients consulted

Step 1. Input the total number of appointments available, \( N \), the no-show rates \( \gamma_1 \) and \( \gamma_2 \), and the joint probability mass function \( p(d_1, d_2) = P(D_1 = d_1, D_2 = d_2) \) of \( D_1 \) and \( D_2 \) for \( 0 \leq d_1 \leq N \) and \( 0 \leq d_2 \leq N \).

Step 2. Calculate \( p_1(d_1) = P(D_1 = d_1), p_2(d_2) = P(D_2 = d_2), F_1(d_1) = P(D_1 \leq d_1), F_2(d_2) = P(D_2 \leq d_2) \), and \( F(d_1, d_2) = P(D_1 \leq d_1, D_2 \leq d_2) \) for \( 0 \leq d_1 \leq N \) and \( 0 \leq d_2 \leq N \).

Step 3. Let \( i = 0 \). Calculate \( Q(i) = (1 - \gamma_2)[N - \sum_{d_2=0}^{N} (N - d_2)p_2(d_2)] \).

Step 4. Let \( i = i + 1 \). Calculate

\[
Q(i) = Q(i - 1) - (\gamma_1 - \gamma_2)[1 - F_1(i - 1)] + (1 - \gamma_2)[F_2(N - i) - F(i - 1, N - i)].
\]

Step 5. If \( i = N \), stop; otherwise, go to Step 4.

It is obvious that as the total number of appointments available, \( N \), increases, the computation effort of Procedure 1 increases. Fortunately, in most outpatient clinics in the United States, the total number of appointments that can be scheduled with a provider in a clinic session only ranges from several to about 30. Therefore, we can compute all \( Q(N_1) \) for all \( N_1 \in [0, N] \) using Procedure 1, and then easily find the optimal solution, \( N^*_1 \), which maximizes the expected number of patients consulted.

5. Numerical examples

Since the demand for prescheduled or open-access appointments is the number of requests for appointments occurring in a given time period, a Poisson distribution is a routine choice for the demand. In numerical examples, the demands for prescheduled and open-access appointments are assumed to have two independent Poisson distributions or a bivariate Poisson distribution with positive correlation. Before describing the numerical examples, we first briefly introduce the bivariate Poisson distribution with positive correlation.

5.1. Bivariate Poisson distribution (Johnson et al., 1997)

Suppose that \( Y_1, Y_2, \) and \( Y_{12} \) are mutually independent Poisson random variables with means \( \theta_1, \theta_2, \) and \( \theta_{12} \), respectively. Let \( X_1 = Y_1 + Y_{12} \), and \( X_2 = Y_2 + Y_{12} \). Then \( X_1 \) and \( X_2 \) are dependent random variables, which have jointly a bivariate Poisson distribution. The marginal distributions of \( X_1 \) and \( X_2 \) are Poisson distributions with means \( \theta_1 + \theta_{12} \) and \( \theta_2 + \theta_{12} \). The covariance between \( X_1 \) and \( X_2 \) is \( \theta_{12} \), and the correlation coefficient between \( X_1 \) and \( X_2 \) is

\[
\rho(X_1, X_2) = \frac{\theta_{12}}{\sqrt{(\theta_1 + \theta_{12})(\theta_2 + \theta_{12})}}.
\]  

(23)
Since \( \theta_{12} \) is positive, the correlation between \( X_1 \) and \( X_2 \) is positive. The joint probability mass function of \( X_1 \) and \( X_2 \) can be computed using the recurrence relations

\[
\begin{align*}
    x_1 p(x_1, x_2) &= \theta_1 p(x_1 - 1, x_2) + \theta_{12} p(x_1 - 1, x_2 - 1), \\
    x_2 p(x_1, x_2) &= \theta_2 p(x_1, x_2 - 1) + \theta_{12} p(x_1 - 1, x_2 - 1).
\end{align*}
\]

5.2. Experimental design

The numerical examples are chosen to analyze the sensitivity of the optimal percentage of appointments held open for a provider in a clinic session, \((N - N_1)/N\), to the total number of appointments available with the provider in the session, the no-show rates \( \gamma_1 \) and \( \gamma_2 \), and the distribution of the demands for prescheduled and open-access appointments with the provider in the session. In a common 4-hour clinic session with 15-minute appointment slots, there are 16 appointment slots that can be booked with each provider. If one appointment is scheduled with a provider in each appointment slot, the total number of appointments available is 32. If 50% of appointment slots are double-booked (i.e., two appointments are scheduled in two successive appointment slots), the total number of appointments available is 24. In some cases, the total number of appointments available is less than 16 if an appointment is scheduled in two successive appointment slots. Therefore, four levels of the total number of appointments available in Table 1 are tested in the numerical examples.

According to Theorem 1, the optimal percentage of appointments held open depends on the ratio of the show-up rate of prescheduled appointments to open-access appointments, \((1 - \gamma_1)/(1 - \gamma_2)\). The no-show rate increases from about 5% to over 35% when the length of interval from the date an appointment is scheduled to the appointment date increases from within one week to over 10 weeks (Kodjababian, 2003). Thus, four combinations of the no-show rates of prescheduled and open-access appointments in Table 1 are investigated in the numerical examples. The ratio of the two show-up rates, \((1 - \gamma_1)/(1 - \gamma_2)\), is 0.889 for the combinations \((\gamma_1, \gamma_2)\) of \((0.1556, 0.05)\) and \((0.2, 0.1)\), while the ratio is 0.667 for the combinations \((\gamma_1, \gamma_2)\) of \((0.3667, 0.05)\), and \((0.4, 0.1)\).

In the numerical examples, two independent Poisson distributions or a bivariate Poisson distribution with correlation coefficient 0.2 or 0.4 is used to approximate the distribution of demands for prescheduled and open-access appointments. 12 combinations of average demands for prescheduled and open-access appointments are tested in the numerical examples. Table 1 summarizes the total numbers of appointments available, the no-show rates, and the demand distributions in the numerical examples.

The \(Q(N_1)\) of all 576 numerical examples are computed using Procedure 1 in Section 4, and then their optimal solutions are found based on their \(Q(N_1)\) for all \(N_1 \in [0, N]\). Procedure 1 is coded using MATLAB 7.0.4. The computation time to find the optimal solutions to all numerical examples is less than 1 minute on a DELL Pentium IV 2.8G personal computer.

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Experimental design</th>
</tr>
</thead>
<tbody>
<tr>
<td>Factor</td>
<td>Levels</td>
</tr>
<tr>
<td>Total number of appointments available ((N))</td>
<td>12, 16, 24, and 32</td>
</tr>
<tr>
<td>No-show rates of prescheduled and open-access appointments ((\gamma_1, \gamma_2))</td>
<td>(0.1556, 0.05), (0.2, 0.1), (0.3667, 0.05), and (0.4, 0.1)</td>
</tr>
<tr>
<td>Demand distribution (D_1) and (D_2)</td>
<td>Independent Poisson distributions</td>
</tr>
<tr>
<td>Demand distribution (D_1) and (D_2)</td>
<td>Bivariate Poisson distributions with correlation coefficient ((\rho)) 0.2 or 0.4</td>
</tr>
<tr>
<td>Average demands for prescheduled and open-access appointments ((E(D_1), E(D_2)))</td>
<td>(0.16N, 0.64N), (0.4N, 0.4N), (0.64N, 0.16N), (0.2N, 0.8N), (0.5N, 0.5N), (0.8N, 0.2N), (0.24N, 0.96N), (0.6N, 0.6N), (0.96N, 0.24N), (0.4N, 1.6N), (N, N), and (1.6N, 0.4N)</td>
</tr>
</tbody>
</table>
5.3. Results and discussion

For 48 of 576 numerical examples, the optimal percentage of appointments held open is 100%. Table 2 summarizes the capacity of a provider to offer appointments in a session, no-show rates, and demand distributions of the 48 cases. Although the 48 cases include all combinations of four levels of provider capacity ($N$), four levels of no-show rates ($c_1, c_2$), and three levels of correlation, the average demand for open-access appointments is much higher than the provider capacity in all 48 cases. This result reveals that when the average demand for open-access appointments is much higher than provider capacity, the optimal percentage of appointments held open is 100% regardless of the demand for prescheduled appointments, the correlation between the demands for prescheduled and open-access appointments, and the no-show rates of prescheduled and open-access appointments. Therefore, for a provider having a higher demand for open-access appointments than his/her capacity in some clinic sessions, all appointments with the provider in these sessions should be held open if no appointments have to be prescheduled due to clinical necessity.

According to Corollary 1, if the demands for prescheduled and open-access appointments are independent of each other, the optimal percentage of appointments held open does not depend on the demand for prescheduled appointments. Therefore, for the cases with independent demands for prescheduled and open-access appointments, we only investigate the impact of the demand for open-access appointments on the optimal percentage of appointments held open. Fig. 1 demonstrates that for a fixed ratio of the two show-up rates, \((1 - \gamma_1)/(1 - \gamma_2)\), the optimal percentage of appointments held open tends to increase with the increase in the ratio of the average demand for open-access appointments to provider capacity, \(E(D_2)/N\), when \(E(D_2)\) is not higher than provider capacity. According to the conditions for \(N_1^*\) in inequalities (17) and (18), if the demand \(D_2\) is Poisson distributed, the optimal number of open-access appointments, \(N - N_1^*\), is a non-decreasing function of \(E(D_2)\) because for any integer \(i\), \(P(D_2 > i)\) increases as \(E(D_2)\) increases. Meanwhile, Fig. 1 also illustrates that for a fixed ratio \(E(D_2)/N\), the optimal percentage of appointments held open slightly decreases as the ratio \((1 - \gamma_1)/(1 - \gamma_2)\) increases.

In addition, Fig. 1 also reveals that when \(E(D_2)\) is not higher than provider capacity, for a fixed ratio \((1 - \gamma_1)/(1 - \gamma_2)\) there exists an approximately linear relationship between the optimal percentage of appointments held open and the ratio \(E(D_2)/N\). Using a normal distribution to roughly approximate the Poisson distribution for \(D_2\), we have \(N - N_1^* \approx \lfloor E(D_2) + z_\alpha \sqrt{E(D_2)} \rfloor\) from inequalities (17) and (18), where \(\lfloor \cdot \rfloor\) means to round up, \(z_\alpha\) is the \(\alpha\) percentile of the standard normal distribution, and \(z = 1 - (1 - \gamma_1)/(1 - \gamma_2)\). When \(z\) is not far from 0.5, \(z_\alpha \sqrt{E(D_2)}\) is much smaller than \(E(D_2)\). Thus there is an approximate linear relationship

### Table 2
Cases with the optimal percentage of open-access appointments of 100%

<table>
<thead>
<tr>
<th>Provider capacity (N)</th>
<th>No-show rate</th>
<th>Demand</th>
<th>(E(D_1))</th>
<th>(E(D_2))</th>
<th>(\rho (D_1, D_2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>0.1556</td>
<td>0.05</td>
<td>4.8</td>
<td>19.2</td>
<td>0, 0.2, or 0.4</td>
</tr>
<tr>
<td>0.2</td>
<td>0.05</td>
<td>4.8</td>
<td>19.2</td>
<td>0, 0.2, or 0.4</td>
<td></td>
</tr>
<tr>
<td>0.3667</td>
<td>0.05</td>
<td>4.8</td>
<td>19.2</td>
<td>0, 0.2, or 0.4</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.1</td>
<td>4.8</td>
<td>19.2</td>
<td>0, 0.2, or 0.4</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>0.1556</td>
<td>0.05</td>
<td>6.4</td>
<td>25.6</td>
<td>0, 0.2, or 0.4</td>
</tr>
<tr>
<td>0.2</td>
<td>0.05</td>
<td>6.4</td>
<td>25.6</td>
<td>0, 0.2, or 0.4</td>
<td></td>
</tr>
<tr>
<td>0.3667</td>
<td>0.05</td>
<td>6.4</td>
<td>25.6</td>
<td>0, 0.2, or 0.4</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.1</td>
<td>6.4</td>
<td>25.6</td>
<td>0, 0.2, or 0.4</td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>0.1556</td>
<td>0.05</td>
<td>9.6</td>
<td>38.4</td>
<td>0, 0.2, or 0.4</td>
</tr>
<tr>
<td>0.2</td>
<td>0.05</td>
<td>9.6</td>
<td>38.4</td>
<td>0, 0.2, or 0.4</td>
<td></td>
</tr>
<tr>
<td>0.3667</td>
<td>0.05</td>
<td>9.6</td>
<td>38.4</td>
<td>0, 0.2, or 0.4</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.1</td>
<td>9.6</td>
<td>38.4</td>
<td>0, 0.2, or 0.4</td>
<td></td>
</tr>
<tr>
<td>32</td>
<td>0.1556</td>
<td>0.05</td>
<td>12.8</td>
<td>51.2</td>
<td>0, 0.2, or 0.4</td>
</tr>
<tr>
<td>0.2</td>
<td>0.05</td>
<td>12.8</td>
<td>51.2</td>
<td>0, 0.2, or 0.4</td>
<td></td>
</tr>
<tr>
<td>0.3667</td>
<td>0.05</td>
<td>12.8</td>
<td>51.2</td>
<td>0, 0.2, or 0.4</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.1</td>
<td>12.8</td>
<td>51.2</td>
<td>0, 0.2, or 0.4</td>
<td></td>
</tr>
</tbody>
</table>
between $N - N^*_1$ and $[E(D_2)]$. The rounding and the error of the normal approximation to Poisson distributions also contribute to the variation of data points around each trend line. On the other hand, when the distance of $\alpha$ from 0.5 increases, the increase in $z_\alpha$ results in the increase in the non-linearity of the relationship between $N - N^*_1$ and $E(D_2)$. This explains that the variance of data points around the trend line for the ratio $(1 - \gamma_1)/(1 - \gamma_2)$ of 0.889 is larger than the variance for the ratio $(1 - \gamma_1)/(1 - \gamma_2)$ of 0.667.

For the cases with dependent demands for prescheduled and open-access appointments, Figs. 2 and 3 depict the impacts of demands $D_1$ and $D_2$ on the optimal percentage of appointments held open, respectively. Fig. 2 illustrates that there is no obvious relationship between the optimal percentage of appointments held open and the ratio of the average demand for prescheduled appointments to the provider capacity, $E(D_1)/N$. Fig. 3 shows that for a fixed ratio $(1 - \gamma_1)/(1 - \gamma_2)$, the optimal percentage of appointments held open tends to increase with the increase in the ratio of the average demand for open-access appointments to provider capacity, $E(D_2)/N$, when $E(D_2)$ is not higher than provider capacity. Using a bivariate normal distribution to roughly approximate the bivariate Poisson distribution for the demands $D_1$ and $D_2$, we obtain $N - N^*_1 \approx [E(D_2) + z_\beta \sqrt{E(D_2)}]$ from inequalities (9) and (10), where $z_\beta$ is the $\beta$ percentile of the standard normal distribution, and $\beta$ is a function of $N^*_1$ and $E(D_1)$ as
In Eq. (26), $\Phi(*)$ denotes the cumulative probability distribution function of the standard normal distribution, and

$$P(D_1 \leq N'_1 - 1, D_2 \leq N - N'_1) = P \left( X^2 \leq \frac{1}{1-\rho} \left[ \left( 1 - \frac{N'_1 - 1 - E(D_1)}{\sqrt{E(D_1)}} \right)^2 - 2\rho \frac{(N'_1 - 1 - E(D_1))(N - N'_1 - E(D_2))}{\sqrt{E(D_1)E(D_2)}} + \left( 1 - \frac{N - N'_1 - E(D_2)}{\sqrt{E(D_2)}} \right)^2 \right] \right),$$

where $X^2$ is the $\chi^2$ distribution, and $\rho$ is the correlation coefficient between $D_1$ and $D_2$. Although the change in $E(D_2)$ indirectly affects $N - N'_1$ through its impact on $\beta$, its impact on $N - N'_1$ is much less than the impact of $E(D_2)$ on $N - N'_1$. Therefore, even for dependent demands, the optimal percentage of appointments held open depends more on the demand for open-access appointments than on the demand for prescheduled appointments.

Fig. 4 elaborates on Figs. 1–3 to depict the impact of the correlation between demands for prescheduled and open-access appointments on the optimal percentage of appointments held open. For the fixed ratios $(1 - \gamma_1)/(1 - \gamma_2)$ and $E(D_2)/N$, the change in the correlation between the two demands very slightly changes the optimal percentage of appointments held open.

In summary, for both independent and dependent demands for prescheduled and open-access appointments, the optimal percentage of appointments held open is mainly determined by the ratio $E(D_2)/N$ and the ratio $(1 - \gamma_1)/(1 - \gamma_2)$ when is not higher than provider capacity. When $E(D_2)$ is not higher than provider
6. Conclusions and future research

This paper presents the first closed-form approach to determine the optimal percentage of appointments held open in a clinic session, which is one of the critical factors for clinical administrators to successfully implement advanced access scheduling. The conditions for determining the optimal percentage of appointments held open and the procedure to find the optimal percentage are presented. The conditions show that the optimal percentage of open-access appointments with a provider in a session depends on the capacity of the provider, the ratio of the show-up rate of prescheduled appointments to the show-up rate of open-access appointments, and the joint distribution of demands for prescheduled and open-access appointments.

This paper investigates the sensitivity of the optimal percentage of appointments held open for a provider in a clinic session to provider capacity, the no-show rates of prescheduled and open-access appointments, and the distribution of demands for prescheduled and open-access appointments.

Fig. 4. Impact of the correlation between two demands: (a) $(1 - \gamma_1)/(1 - \gamma_2) = 0.667$ and (b) $(1 - \gamma_1)/(1 - \gamma_2) = 0.889$.

capacity, all appointments should be held open if no appointments have to be prescheduled due to clinical necessity.

This paper presents the first closed-form approach to determine the optimal percentage of appointments held open in a clinic session, which is one of the critical factors for clinical administrators to successfully implement advanced access scheduling. The conditions for determining the optimal percentage of appointments held open and the procedure to find the optimal percentage are presented. The conditions show that the optimal percentage of open-access appointments with a provider in a session depends on the capacity of the provider, the ratio of the show-up rate of prescheduled appointments to the show-up rate of open-access appointments, and the joint distribution of demands for prescheduled and open-access appointments.
is not higher than provider capacity, the optimal percentage of appointments held open mainly depends on the ratio of his/her average demand for open-access appointments to his/her capacity and the ratio of the show-up rates for prescheduled and open-access appointments. Our results also demonstrate that for the fixed ratio of average demand for open-access appointments to provider capacity and fixed ratio of the two show-up rates, different correlations between the two demands very slightly affect the optimal percentage of appointments held open. This will greatly help the clinical administrators in implementing advanced access scheduling in that the estimation of demand distributions can be significantly simplified by not considering the dependency between the two demands.

Like the average number of patients consulted by each provider in a clinic session, the variance of the number of patients consulted by each provider in a clinic session is also a measure for the match of daily demand and capacity. For future research, the quantitative approach can be extended by considering both the expectation and variance of the number of patients consulted by a provider in a session. In addition, to accurately determine the optimal percentage of appointments held open for each provider in each clinic session using the approach presented, a model is needed to predict the demands for prescheduled and open-access appointments with each provider in each session.

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The authors want to thank the administrators, physicians, and staff in the Indiana University Medical Group – Primary Care clinics for providing insights into traditional and advanced access appointment scheduling in outpatient clinics. The first author is grateful to Professor Bruce W. Schmeiser for the valuable discussions and references. The authors gratefully acknowledge the support from the Regenstrief Center for Healthcare Engineering.

Appendix. Change in the expected number of patients consulted in a clinic session

When the maximal number of appointments that can be prescheduled with a provider in a session increases from \( N_1 \) to \( N_1 + 1 \), according to Eq. (8), the change in the expected number of patients consulted by the provider in the session is

\[
Q(N_1 + 1) - Q(N_1) = -\gamma_1(N_1 + 1) - \sum_{d_1=0}^{N_1+1} (N_1 + 1 - d_1)p_1(d_1)
\]

\[
+ (1 - \gamma_2) \left[ \sum_{d_1=0}^{N_1+1} \sum_{d_2=0}^{N_1} (N - d_1 - d_2)p(d_1, d_2) - \sum_{d_1=N_1+2}^{\infty} \sum_{d_2=0}^{\infty} (N - N_1 - 1 - d_2)p(d_1, d_2) \right]
\]

\[
+ (\gamma_1 - \gamma_2) \left[ N_1 - \sum_{d_1=0}^{N_1} (N_1 - d_1)p_1(d_1) \right]
\]

\[
- (1 - \gamma_2) \left[ \sum_{d_1=0}^{N_1} \sum_{d_2=0}^{N_1} (N - d_1 - d_2)p(d_1, d_2) - \sum_{d_1=N_1+1}^{\infty} \sum_{d_2=0}^{\infty} (N - N_1 - d_2)p(d_1, d_2) \right]
\]

\[
= -\gamma_1(N_1 + 1) - (1 - \gamma_2) \left[ \sum_{d_1=0}^{N_1+1} \sum_{d_2=0}^{N_1} (N - d_1 - d_2)p(d_1, d_2) - \sum_{d_1=N_1+2}^{\infty} \sum_{d_2=0}^{\infty} (N - N_1 - 1 - d_2)p(d_1, d_2) \right]
\]

\[
- (1 - \gamma_2) \left[ \sum_{d_1=0}^{\infty} \sum_{d_2=0}^{N_1} (N - N_1 - 1 - d_2)p(d_1, d_2) - \sum_{d_1=N_1+2}^{\infty} \sum_{d_2=0}^{\infty} (N - N_1 - d_2)p(d_1, d_2) \right]
\]

\[
= -\gamma_1(N_1 + 1) + (\gamma_1 - \gamma_2) \sum_{d_1=0}^{N_1} p_1(d_1) - (1 - \gamma_2) \sum_{d_2=0}^{\infty} (N - N_1 - 1 - d_2)p(N_1 + 1, d_2)
\]
\[
(1 - \gamma_2) \left[ \sum_{d_1=N_1+1}^{\infty} \sum_{d_2=0}^{N_1} p(d_1, d_2) + \sum_{d_2=0}^{N_1} (N - N_1 - d_2)p(N_1 + 1, d_2) \right]
\]

\[
= -\gamma_1 \gamma_2 + \gamma_1 \sum_{d_1=N_1+1}^{\infty} p_1(d_1) + (1 - \gamma_2) \sum_{d_2=0}^{N_1} \sum_{d_1=N_1+1}^{\infty} p(d_1, d_2)
\]

\[
= -\gamma_1 \gamma_2 + \gamma_1 [1 - F_1(N_1)] + (1 - \gamma_2) [F_2(N - N_1 - 1) - F(N_1, N - N_1 - 1)]
\]

\[
= [1 - \gamma_1] [1 - F_1(N_1)] - (1 - \gamma_2) [F_1(N_1) - F_2(N - N_1 - 1) + F(N_1, N - N_1 - 1)], \quad (A.1)
\]

where \( F_1(i) = P(D_1 \leq i) \) is the cumulative probability distribution function of \( D_1 \), \( F_2(i) = P(D_2 \leq i) \) is the cumulative probability distribution function of \( D_2 \), and \( F(i, j) = P(D_1 \leq i, D_2 \leq j) \) is the joint cumulative probability distribution function of \( D_1 \) and \( D_2 \).

References


